

# Complex Wave Shaping

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## Abstract

This paper discusses modulating techniques for sound synthesis using mappings of the complex unit circle. It is an exploration of some common topics in music dsp translated to the dual issues in (periodic) oscillator design.

## 1 Introduction

We will investigate the use of complex functions as a modulators, oscillators or “wave shapers” to produce sound with timbral control. The methods introduced here are not new. This paper is mainly intended as an illustration of the elegance of working with complex signals, as apposed to the more cumbersome real approach with sines and cosines, and the duality between designing continuous oscillators and waveshapers versus discrete signals and filters.

Let’s introduce some of the players. The basic signal we use to generate waveforms is the exponential map of the imaginary axis, or complex oscillation  $x(t) = e^{j\omega t}$ . For analytic functions  $f(z)$  we will use the power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (1)$$

for analyzing the effect of a waveshaper  $f(z)$  on  $x(t)$ , since its coefficients  $a_k$  represent the harmonic content of the resulting signal.

This is almost the same as the  $z$ -transform, except for the sign flip. What we will do most of the time is use the duality between time and frequency analysis. In discrete filter analysis, the spectrum is continuous and periodic and lives on the complex unit

circle. Filters are infinite length discrete sequences (discrete impulse responses). Here we turn the picture around: signals live on the complex unit circle and are periodic and continuous. Spectra are discrete (harmonic) infinite series. I.e. rational waveshaping functions are the equivalent of finite order discrete IIR linear filters, which are “spectrum waveshapers”. Finite waveshaping polynomials are in the same sense equivalent to discrete FIR filters.

The function  $\frac{z}{|z|}$ , which maps the complex plane onto the unit circle, can be used as a general purpose (nonlinear) normalizing function. Since it is not differentiable, it is not an analytic function and does not have a power series expansion, so it is not very useful for analysis and should be used with care.

The function

$$f_{a,\theta}(z) = e^{j\theta} \frac{z - a}{1 - \bar{a}z} \quad (2)$$

with  $z, a \in \mathbb{C}$ ,  $\theta \in \mathbb{R}$  and  $|a| \neq 1$  is an invertible map from the unit circle onto itself. If  $|z| = 1$  then  $|f_{a,\theta}(z)| = 1$ . The  $f$  form a group with composition the group operation. The inverse element of  $f_{a,\theta}$  is  $f_{-a,-\theta}$ . We will have another look at this function next and relate it to the DSF. In the dual representation, this is of course the  $z$  transform of a discrete all-pass filter. The more general fractional linear transform can be used as well. The dual being a complex one pole/zero filter.

## 2 Conformal maps and DSF

We set  $\theta = 0$  in the following because it will only amount to a constant a rotation. Let  $a = rz'$  with  $|z'| = 1$ . Suppose  $z$  and  $z'$  are two unit norm complex

oscillators defined by  $z(t) = e^{j\alpha t}$  and  $z'(t) = e^{j\beta t}$ . Our modulator driven by the two oscillators has the form

$$\frac{z - rz'}{1 - rz'z}. \quad (3)$$

The signal generated is

$$s(t) = \frac{e^{j\alpha t} - re^{j\beta t}}{1 - re^{j(\alpha-\beta)t}} \quad (4)$$

and is periodic when  $\frac{\alpha}{\beta}$  is rational. The constant  $r$  can be seen as a modulation index. We write this in a more useful way as

$$s(t) = e^{j\alpha t} \frac{1 - re^{-j(\alpha-\beta)t}}{1 - re^{j(\alpha-\beta)t}}. \quad (5)$$

We see that the factor  $e^{j\alpha t}$  is just a frequency shift and the second term only depends on  $\delta = \alpha - \beta$ . We can write the denominator of (5) as power series

$$\frac{1}{1 - re^{j\delta t}} = \sum_{k=0}^{\infty} r^k e^{jk\delta t}. \quad (6)$$

This expansion is of course the limit case of the discrete summation formula (DSF)

$$\frac{1 - re^{jN\delta t}}{1 - re^{j\delta t}} = \sum_{k=0}^{N-1} r^k e^{jk\delta t}. \quad (7)$$

which is the dual of truncated IIR filters. This establishes the link between conformal map modulation and exponentially damped harmonic spectra. The more general case of fractional linear transforms still has a discrete spectrum.

The spectrum of  $s(t)$  is thus harmonic with spacing  $\delta$ , except for the frequency shift, and exponentially damped

$$s(t) = e^{j\alpha t} (re^{-j\delta t} + (1 - r^2) \sum_{k=0}^{\infty} r^k e^{jk\delta t}). \quad (8)$$

### 3 Power series

Let's work with (6) as our main representation some more, and abandon the unit norm output assumption for a while. If we are interested in bandlimited synthesis, i.e. to prevent aliasing, we can always obtain a finite spectrum by switching to (7).

An odd harmonic series can be synthesized using

$$\frac{e^{j\delta t}}{1 - re^{j2\delta t}} = \sum_{k=0}^{\infty} r^k e^{j(2k+1)\delta t}. \quad (9)$$

Several variants can be constructed analogous to this, creating shifted harmonic, formant like waves. The method is related to FM, only the spectrum is one sided, exponentially damped and can be made finite. A drawback is that formants cannot be shifted independently from pitch in a continuous way. This can be solved by using a function like

$$c(\alpha, x) = ae^{jn\delta t} + (1 - a)e^{j(n+1)\delta t} \quad (10)$$

As the carrier wave. Here  $n = [x]$ , the integer part of  $x$  and  $a = x - n$ , the fractional part of  $x$ , as proposed in [1]. In order for this method to work, one must take care to ensure phase locking between the two oscillator terms. This can pose a problem when a time varying oscillator is implemented, since due to the locking, a change in frequency will cause a phase jump in at least one of the locked oscillators. This can be solved by smoothing out the discontinuous frequency change, but not without additional frequency modulation. With locked oscillators, the frequency can only be changed without a phase discontinuity if it is done when the phase of the basic harmonic is 0.

This method is dual to fractional delay using linear interpolation. In fact, the discrete spectrum is resampled and shifted fractionally. This method can of course be generalized to other forms of interpolation and filtering.

Squaring (6) is equivalent to convolving the spectral envelope with itself. This gives a more localized formant structure. Taking the real part of  $f$  turns the harmonic spectrum into a symmetric one that can be shifted up (or down) in frequency to build a formant

like symmetric wave packet<sup>1</sup>. Squaring the real signal before multiplication gives a smoother formant packet.

If  $r$  is complex, we can introduce a phase shift in the harmonics. Combining both  $r$  and  $\bar{r}$  we can construct damped sinusoidal spectral envelopes. This enables us to use standard sinusoidal methods to model a spectral envelope. If  $f_{r,\delta}(t)$  represents (6) or (7) we see that  $f_{\bar{r},\delta}(t) = \overline{f_{r,-\delta}(t)}$ .

Normalization can sometimes be desirable to get a constant power output over the modulation range<sup>2</sup>. If we define the inner product between two periodic signals  $s(t)$  and  $s'(t)$  with common period  $T$  to be  $\langle s, s' \rangle P_p = \frac{1}{T} \int_0^T \overline{s'(t)} s(t) dt$  and the power to be  $P = \langle s, s \rangle$  and note that the components of (7) are orthogonal, we can see the power is equal to

$$\sum_{k=0}^{N-1} |r|^{2k} = \frac{1 - |r|^{2N}}{1 - |r|^2} \quad (11)$$

## 4 Fractional spectrum shift

Expression (10) is an interesting one to take a closer look at. It is a carrier signal that can be used to shift a harmonic spectrum without disturbing its harmonicity. In the same way that a fractional delay for discrete signals can be considered a bit strange, this function is periodic but seems to behave like a single, localized oscillation, not related to this period.

In fact, it's not at all a single localized oscillation but a periodic wave with a very narrow bandwidth. The bandwidth depends on  $a$ . We saw before that the most straightforward way to implement a time varying version of this oscillator is to only make frequency jumps when the base oscillator  $e^{j\delta t}$  is at zero phase. Since updates then occur at the period of the

<sup>1</sup>Combined with (10) this is essentially the Phase Aligned Formant (PAF) method

<sup>2</sup>When using a "squashed" normalized DSF and allowing the modulation index to move beyond unit magnitude, we get a very nice sounding broad spectrum transient when  $r$  passes through  $|r| = 1$ . Especially so when the change in  $r$  is not made instantaneous, but i.e. processed by a lowpass filter (deziped). During a fast change in  $r$ , the DSF normalization (which limits the number of harmonics to  $N$ ) is not valid. This gives a large signal which then will be squashed by the normalizing function.

waveform, smooth frequency shifts are still attainable.

The quality of this interpolation can be made larger when using more terms. A straightforward approach is to use polynomial interpolation to construct a better interpolating filter. Just like a polynomial fractional delay filter that preserves DC needs to have a zero at Nyquist to introduce the 180 degree phase discontinuity when  $a = 0.5$ , our "parametric polynomial" acting as waveshaper on the unit circle needs to vanish for some point on the unit circle to "disguise" the phase discontinuity. When  $a = 0$ , there is no vanishing point. The switching between the two regimes when the center frequency is varied will cause some amplitude modulation. This amplitude modulation is characteristic of all polynomial interpolators.

We can trade in some bandwidth of the modulator, amounting to a waveshaping function that will have a smaller magnitude on some parts of the unit circle, in order to decrease this modulation. This is equivalent to resampling using bandlimited interpolation.

So it is not clear what would be the preferred route. The problem is completely analogous to signal resampling, but the trade-offs are different. The time varying filtering effect of polynomial interpolators (amplitude modulation in our case) is usually not a problem for a high order interpolator, since the artifacts are high frequency. In our case all modulation will be perceivable so it might be more interesting to trade in some more bandwidth (allow the waveshaper to approach zero over a longer part of the circle) than to use a high order interpolator. However, when using a polynomial interpolator, the part of the waveshaper that will be explicitly time varying, introducing the amplitude modulation, will be smaller when the order increases.

## 5 More unrelated stuff

The space spanned by the  $f_{a,\theta}$  is isomorphic to the space of matrices of the form

$$m'_{a,\theta} = e^{j\theta} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}. \quad (12)$$

$|a| \neq 1$  ensures  $\det m'_{a,\theta} \neq 0$ . Dividing the nomi-

nator and denominator of (2) by  $\sqrt{1 - |a|^2}$  does not change the value of  $f_{a,\theta}$  so we will let the matrix with unit norm determinant

$$m_{a,\theta} = \frac{e^{j\theta}}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}. \quad (13)$$

correspond to  $f_{a,\theta}$ . The set of  $m_{a,\theta}$  is a group under multiplication. The mapping between  $f$  and  $m$  is a group homomorphism mapping composition of  $f$  to multiplication of  $m$ . The inverse of  $m_{a,\theta}$  is given by  $m_{-a,-\theta}$ . This means that  $f_{a,\theta} \circ f_{-a,-\theta}(z) = z$ . The  $m_{a,\theta}$  span a hyperbolic space where the inverse is given by the anti-hermitian of  $m$ .

For the angular representation We will write  $a = re^{j\beta}$ . For  $0 < |r| < 1$  we can set  $r = \sin \gamma$ . Transforming (2) we get

$$f_{\beta,\gamma}(z) = \frac{z + \sin \theta e^{j\beta}}{1 + \sin \theta e^{-j\beta} z}. \quad (14)$$

Transforming (13) in the same way we get

$$m_{\gamma,\beta} = \frac{1}{\cos \gamma} \begin{pmatrix} 1 & re^{j\beta} \\ re^{-j\beta} & 1 \end{pmatrix}. \quad (15)$$

Writing (16) as a function of  $\alpha$  with  $z = e^{j\alpha}$  we get

$$f(\alpha, \beta, \gamma) = \frac{e^{j\alpha} + \sin \gamma e^{j\beta}}{1 + \sin \gamma e^{-j\beta} e^{j\alpha}}. \quad (16)$$